Operatorial subordination in free probability and loop products of graphs

Romuald Lenczewski

Department of Mathematics and Computer Science
Wroclaw University of Technology

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Topics

1. analytic subordination for $\mu \boxplus \nu$ and $\mu \boxtimes \nu$
2. subordination in terms of s-free convolutions
3. 'complete' decompositions of $\mu \boxplus \nu$ and $\mu \boxtimes \nu$
4. operatorial subordination
5. product graphs $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$ for independence $\mathcal{I}$
6. addition theorem for $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$
7. loop products of graphs $\mathcal{G}_1 \mathcal{I}_{\circlearrowright} \mathcal{G}_2$
8. multiplication theorem for $\mathcal{G}_1 \mathcal{I}_{\circlearrowright} \mathcal{G}_2$

Addition

Aspects of addition of noncommutative random variables:

1. independence $\mathcal{I}$
2. addition of random variables $X + Y$
3. additive convolution $\mu +_{\mathcal{I}} \nu$
4. product of graphs $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$
5. Addition Theorem: moments of $\mu +_{\mathcal{I}} \nu = \#$ (walks in $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$)

Transforms

Let μ – probability measure on $\mathbb{R}$ and $z \in \mathbb{C}^+$. Useful transforms:

1. Cauchy transform

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{\mu(dx)}{z-x} = \sum_{n=0}^{\infty} \mu(X^n) z^{-n-1} \ \ (\text{if } \mu \text{ has moments})$$

2. Reciprocal Cauchy transform

$$F_\mu(z) = \frac{1}{G_\mu(z)}$$

3. K-transform

$$K_\mu(z) = z - F_\mu(z)$$

Analytic subordination for $\mu \boxplus \nu$

Theorem [Voiculescu 1993 -compact, Biane 1998 -general]

For probability measures μ, ν on $\mathbb{R}$, it holds that

$$F_{\mu \boxplus \nu}(z) = F_{\mu}(F_1(z)) = F_{\nu}(F_2(z))$$

where $z \in \mathbb{C}_+$ and F_1, F_2 are reciprocal Cauchy transforms of some probability measures on $\mathbb{R}$.

Convolution $\mu \boxright \nu$

Definition

The functions F_1 and F_2 define unique probability measures on $\mathbb{R}$. Therefore, we propose to introduce a binary operation $\boxright$ on $\mathcal{M}_{\mathbb{R}}$, namely

$$F_1(z) = F_{\nu \boxright \mu}(z) \quad \text{and} \quad F_2(z) = F_{\mu \boxright \nu}(z).$$

The convolution $\mu \boxright \nu$ ('half' of $\mu \boxplus \nu$) will be called the *s-free additive convolution*.

Proposition

Subordination equations can then be written in terms of s-free convolutions:

$$\mu \boxplus \nu = \mu \rhd (\nu \boxvdash \mu) \ = \ \nu \rhd (\mu \boxvdash \nu)$$

where $\rhd$ – monotone additive convolution since we have $F_{\mu \rhd \nu}(z) = F_{\mu}(F_{\nu}(z))$.

Freeness with subordination

Assumptions:

1. $(\mathcal{A}, \varphi, \psi)$ – unital algebra with a pair of linear normalized functionals
2. $\mathcal{A}_1$ – unital subalgebra of $\mathcal{A}$
3. $\mathcal{A}_2$ – non-unital subalgebra with an 'internal' unit 1_2, i.e. $1_2 b = b = b 1_2$ for every $b \in \mathcal{A}_2$.

Freeness with subordination

Definition

The pair $(\mathcal{A}_1, \mathcal{A}_2)$ is *free with subordination*, or *s-free*, with respect to (φ, ψ), where $\psi(1_2) = 1$, if
(ii) $\varphi(a_1 a_2 \ldots a_n) = 0$ whenever $a_j \in \mathcal{A}^0_{i_j}$ and $i_1 \neq i_2 \neq \ldots \neq i_n$
(ii) $\varphi(w_1 1_2 w_2) = \varphi(w_1 w_2) - \varphi(w_2)\varphi(w_2)$ for any $w_1, w_2 \in \mathrm{alg}(\mathcal{A}_1, \mathcal{A}_2)$,

where $\mathcal{A}^0_1 = \mathcal{A}_1 \cap \ker\varphi$ and $\mathcal{A}^0_2 = \mathcal{A}_2 \cap \ker\psi$.

Orthogonal additive convolution

Definition

Let μ and ν be probability measures. The *orthogonal additive convolution* is defined by the reciprocal Cauchy transform

$$F_{\mu \vdash \nu}(z) = F_{\mu}(F_{\nu}(z)) - F_{\nu}(z) + z$$

Equivalently,

$$K_{\mu \vdash \nu}(z) = K_{\mu}(F_{\nu}(z)) = K_{\mu}(z - K_{\nu}(z))$$

Orthogonal independence

Definition

Let $(\mathcal{A}, \varphi, \psi)$ be unital algebra with a pair of linear normalized functionals and let $\mathcal{A}_1, \mathcal{A}_2$ be non-unital subalgebras of $\mathcal{A}$. We say that $\mathcal{A}_2$ is *orthogonal* to $\mathcal{A}_1$ with respect to (φ, ψ) if

$$\begin{aligned} \varphi(bw_2) &= \varphi(w_1 b) = 0 \\ \varphi(w_1 a_1 b a_2 w_2) &= \psi(b)\varphi(w_1 a_1 a_2 w_2) \\ &- \psi(b)\varphi(w_1 a_1)\varphi(a_2 w_2) \end{aligned}$$

for any $a_1, a_2 \in \mathcal{A}_1$, $b \in \mathcal{A}_2$ and any elements w_1, w_2 of the unital algebra $\mathrm{alg}(\mathcal{A}_1, \mathcal{A}_2)$.

Theorem [R.L. 2006]

If $\mu, \nu \in \mathcal{M}_{\mathbb{R}_+}$ are compactly supported, then we have 'complete' decompositions

$$
\begin{aligned}
\mu \,\boxed{\vdash}\, \nu &= \mu \vdash (\nu \vdash (\mu \vdash (\nu \vdash \ldots)))) \\
\mu \boxplus \nu &= \mu \rhd (\nu \vdash (\mu \vdash (\nu \vdash \ldots))))
\end{aligned}
$$

where the right hand side is understood as the weak limit.

Corollary

The transforms of $\mu \boxvdash \nu$ and $\mu \boxplus \nu$ can be written in the 'continued composition form'

$$
\begin{aligned}
K_{\mu \boxvdash \nu}(z) &= K_{\mu}(z - K_{\nu}(z - K_{\mu}(\ldots))) \\
F_{\mu \boxplus \nu}(z) &= F_{\mu}(z - K_{\nu}(z - K_{\mu}(\ldots)))
\end{aligned}
$$

where the right-hand side is understood as the uniform limit on compact subsets of $\mathbb{C}_{+}$.

Theorem [R.L. 2006]

If $X, Y \in \mathcal{B}(\mathcal{H})$ are free and self-adjoint, where $\mathcal{H} = \mathcal{H}_1 * \mathcal{H}_2$, then one can construct self-adjoint operators $s, S \in \mathcal{B}(\mathcal{H})$ such that
(i) $X + Y = s + S$
(ii) φ–distribution of s is μ
(iii) φ–distribution of S is $\nu \boxminus \mu$
(iv) (s, S) is monotone independent w.r.t. φ.
Moreover, there exists a sequence of orthogonally independent operators corresponding to the 'complete' decomposition of $\mu \boxplus \nu$.

Additive convolutions versus $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$

We have the following correspondence between additive convolutions or probability measures on the real line and products of rooted graphs"'

1. monotone - $\mu \triangleright \nu$ - comb product
2. boolean - $\mu \uplus \nu$ - star product
3. free - $\mu \boxplus \nu$ - free product
4. orthogonal - $\mu \vdash \nu$ - orthogonal product
5. s-free - $\mu \boxvdash \nu$ - s-free product

Addition theorem

Theorem [Accardi, R.L., Sałapata 2006]

Let $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$ be naturally colored and let μ, ν be spectral distributions of $\mathcal{G}_1, \mathcal{G}_2$. If $\mathcal{I}$ stands for monotone, boolean, orthogonal, s-free and free, and Z is the adjacency matrix of $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$, then

$$\varphi(Z^n) = M_{\mu +_{\mathcal{I}} \nu}(n) = |W_n(e)|.$$

where $W_n(e)$ is the set of rooted walks on $\mathcal{G}_1 \mathcal{I} \mathcal{G}_2$ of length n. Moreover, if $Z = S_1 + S_2$ is the decomposition induced by the coloring, then (S_1, S_2) is $\mathcal{I}$-independent.

Transforms

Let μ be a probability measure on $\mathbb{R}_+ = [0, \infty)$ and $z \in \mathbb{C} \backslash \mathbb{R}_+$.
Useful transforms for multiplicative convolutions:

1. $$\psi_\mu(z) = \int_{\mathbb{R}_+} \frac{zt}{1 - zt} d\mu(t) = \sum_{n=1}^{\infty} \mu(X^n) z^n$$

2. $$\eta_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}$$

3. $$\rho_\mu(z) = \frac{\eta_\mu(z)}{z}$$

Analytic subordination for $\mu \boxtimes \nu$

Theorem [Biane 1998]

For probability measures μ, ν on $\mathbb{R}_+$ it holds that

$$\eta_{\mu \boxtimes \nu} = \eta_\mu(\eta_1(z)) = \eta_\nu(\eta_2(z))$$

for $z \in \mathbb{C} \setminus \mathbb{R}_+$, where η_1, η_2 are η-transforms of some probability measures on $\mathbb{R}_+$.

Convolution $\mu \boxslash \nu$

Definition

The functions η_1 and η_2 define unique probability measures on $\mathbb{R}_+$ which are not concentrated at zero. This gives a binary operation $\boxslash$ on $\mathcal{M}_{\mathbb{R}_+} \setminus \{\delta_0\}$, namely

$$\eta_1(z) = \eta_{\nu \boxslash \mu}(z) \quad \text{and} \quad \eta_2(z) = \eta_{\mu \boxslash \nu}(z).$$

The convolution $\mu \boxslash \nu$ is called the *s-free multiplicative convolution*.

Subordination reformulated

Proposition

The subordination equations can be written in terms of s-free multiplicative convolutions as

$$\mu \boxtimes \nu = \mu \circlearrowright (\nu \boxslash \mu) = \nu \circlearrowright (\mu \boxslash \nu)$$

since the monotone multiplicative convolution (Bercovici) is defined by the equation

$$\eta_{\mu \circlearrowright \nu}(z) = \eta_{\mu}(\eta_{\nu}(z))$$

for $z \in \mathbb{C} \backslash \mathbb{R}_{+}$ and $\mu, \nu \in \mathcal{M}_{\mathbb{R}_{+}}$.

Decomposition of $\mu \boxslash \nu$ and $\mu \boxtimes \nu$

Theorem [R.L. 2007]

For compactly supported $\mu, \nu \in \mathcal{M}_{\mathbb{R}_+}$, we have complete decompositions

$$\begin{aligned} \mu \boxslash \nu &= \mu \angle (\nu \angle (\mu \angle (\nu \angle (\ldots)))), \\ \mu \boxtimes \nu &= \mu \circlearrowright (\nu \angle (\mu \angle (\nu \angle (\ldots)))), \end{aligned}$$

and their transforms can be written in the 'continued composition form':

$$\begin{aligned} \rho_{\mu \boxslash \nu}(z) &= \rho_{\mu}(z\rho_{\nu}(z\rho_{\nu}(z\rho_{\mu}(\ldots)))), \\ \eta_{\mu \boxtimes \nu}(z) &= \eta_{\mu}(z\rho_{\nu}(z\rho_{\mu}(z\rho_{\nu}(\ldots)))), \end{aligned}$$

where the right-hand sides are understood as the uniform limits on compact subsets of $\mathbb{C} \backslash \mathbb{R}_+$.

Theorem [R.L. 2007]

If $X, Y \in \mathcal{B}(\mathcal{H})$ are positive and free, where $\mathcal{H} = \mathcal{H}_1 * \mathcal{H}_2$, then there exist positive z, Z such that

(i) $\sqrt{X}Y\sqrt{X} = \sqrt{z}\, Z\sqrt{z}$

(ii) φ–distribution of z is μ

(iii) φ–distribution of Z is $\nu \boxslash \mu$

(iv) $(z-1, Z-1)$ is monotone independent w.r.t. φ.

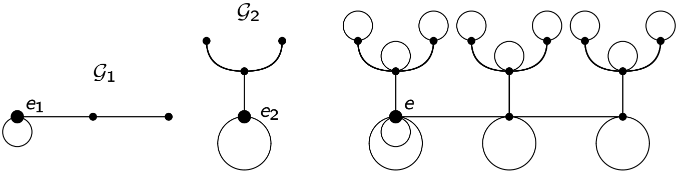
Moreover, there exists a sequence of orthogonally independent operators corresponding to the 'complete' decomposition of $\mu \boxtimes \nu$.

Multiplicative convolutions versus $\mathcal{G}_1 \mathcal{I}_\ell \mathcal{G}_2$

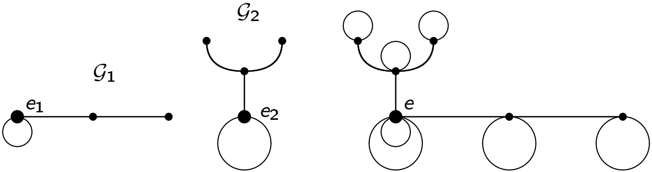
We have the following correspondence:

1. monotone independence - $\mu \circlearrowright \nu$ - comb loop product
2. boolean independence - $\mu \boxast \nu$ - star loop product
3. freeness - $\mu \boxtimes \nu$ - free product
4. orthogonal independence - $\mu \angle \nu$ - orthogonal loop product
5. s-freeness - $\mu \boxslash \nu$ - s-free loop product

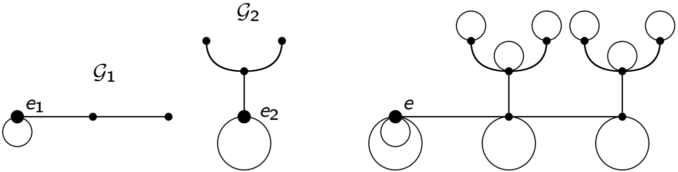
Comb loop product of graphs



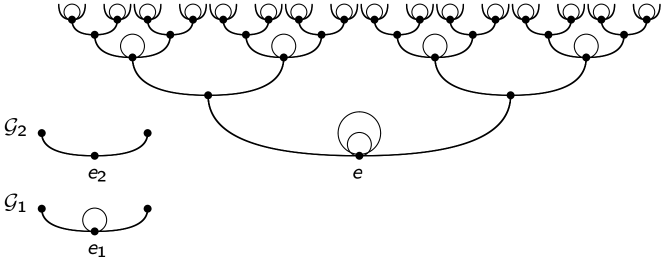
Star loop product of graphs



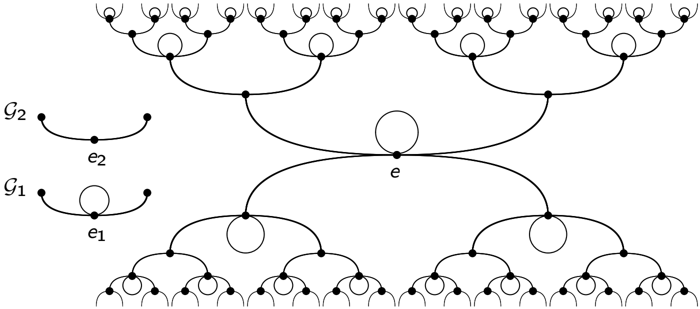
Orthogonal loop product of graphs



s-free loop product of graphs



Free product of graphs



Multiplication theorem

Notations:

1. $\mathcal{G}_1 \mathcal{I}_\ell \mathcal{G}_2$ naturally colored loop product of graphs corresponding to $\mathcal{I}$-independence
2. μ, ν – spectral distributions of $\mathcal{G}_1, \mathcal{G}_2$
3. $D_{2n}(e)$ – rooted alternating double return walks on $\mathcal{G}_1 \mathcal{I}_\ell \mathcal{G}_2$ of length $2n$.

Theorem [R.L. 2007]

If $\mathcal{I}$ refers to monotone, boolean, orthogonal, s-free and free independence, and Z is the adjacency matrix of $\mathcal{G}_1 \mathcal{I}_\ell \mathcal{G}_2$, then

$$N_Z(n) = N_{\mu_1 \times_{\mathcal{I}} \mu_2}(n) = |D_{2n}(e)|.$$

Moreover, if $Z = R_1 + R_2$ is the decomposition induced by the coloring, then $(R_1 - 1, R_2 - 1)$ is $\mathcal{I}$-independent.

References

1. R. Lenczewski, Decompositions of the free additive convolution, J. Funct. Anal. **246** (2007), 330-365.
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3. R. Lenczewski, Operators related to subordination for free multiplicative convolutions, Indiana Univ. Math. J. (2008), to appear.